

When is a pure state of three qubits determined by its single-particle reduced density matrices?

A Sawicki^{1,2}, M Walter³, and M Kuś²

¹*School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK*

²*Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warszawa, Poland*

³*Institute for Theoretical Physics, ETH Zürich, Wolfgang-Pauli-Str. 27, 8093 Zürich, Switzerland*

Using techniques from symplectic geometry, we determine when a pure state of three qubits is up to local unitaries uniquely determined by its reduced density matrices. We moreover show that this is always the case if one is given the additional promise that the quantum state is not convertible to the Greenberger–Horne–Zeilinger (GHZ) state by stochastic local operations and classical communication (SLOCC).

1. Introduction

Finding solutions to Hamilton's equations for a given system is a standard problem in classical mechanics. The system in question is often invariant with respect to a certain group of symmetries, and when the symmetries are continuous, such as in the case of rotational symmetry, they form a Lie group K . It is well understood nowadays that the existence of symmetries is inevitably connected to the first integrals of Hamilton's equations. The celebrated theorem of Arnold (1989) states that in case when $K = T^n$ is an n -dimensional torus and n half the dimension of the phase space M^1 then the system is completely integrable. When the group K is not abelian, it is still possible to find corresponding first integrals, although they typically do not Poisson commute. For example, when a particle is moving in a potential with rotational symmetry, so that $K = SO(3)$, then the conserved quantities are the three components of the angular momentum, corresponding to the invariance of the system with respect to infinitesimal rotations about the axes x , y , and z . These infinitesimal rotations generate the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. There are of course many possible bases of $\mathfrak{so}(3)$, each corresponding to a different choice of rotation axes and each giving three first integrals. The mathematical object which encodes information about the first integrals for all possible choices of generators of $\mathfrak{so}(3)$ is an equivariant map $\mu : M \rightarrow \mathfrak{so}(3)^*$ from the phase space M to the space of linear functionals on the Lie algebra $\mathfrak{so}(3)$. For every infinitesimal symmetry $\xi \in \mathfrak{so}(3)$ one obtains a corresponding first integral by the formula $\mu_\xi(x) = \langle \mu(x), \xi \rangle$, where by $\langle \cdot, \cdot \rangle$ we denote the pairing between linear functionals from \mathfrak{k}^* and vectors

email: Adam.Sawicki@bristol.ac.uk

email: mwalter@itp.phys.ethz.ch

email: marek.kus@cft.edu.pl

¹ A phase space is a symplectic manifold (M, ω) , where ω is a closed nondegenerate two-form.

in \mathfrak{k} . This idea can be generalized to arbitrary Lie groups K and a corresponding map $\mu: M \rightarrow \mathfrak{k}^*$ is called a *momentum map* (Guillemin & Sternberg 1990).

Remarkably, momentum maps appear naturally not only in classical but also in quantum mechanics. Indeed, the Hilbert space \mathcal{H} on which a given quantum-mechanical system is modeled can be seen as a phase space if we identify vectors that differ by a global rescaling or a phase factor $e^{i\phi}$. The set of (pure) quantum states is thus isomorphic to the complex projective space $\mathbb{P}(\mathcal{H})$, which is well-known to be a symplectic manifold. When the considered system consists of N subsystems² then the space \mathcal{H} has the additional structure of a tensor product, namely $\mathcal{H} = \mathcal{H}_1^{\otimes N}$, where \mathcal{H}_1 is the single-particle Hilbert space. The mathematical properties of this structure manifest physically as entanglement, i.e. quantum correlations between subsystems. These correlations are invariant with respect to the local unitary action, i.e. to the action of the Lie group $K = SU(\mathcal{H}_1)^{\times N}$ on \mathcal{H} by the tensor product. Since K preserves the phase space structure of $\mathbb{P}(\mathcal{H})$ one gets a momentum map $\mu: \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{k}^*$, and the image $\mu([v])$ of a quantum state $[v] \in \mathbb{P}(\mathcal{H})$ is, up to some unimportant shifting, the collection of its one-body reduced density matrices (see Section 4). The matrices ρ_i can be diagonalized by the action of K and their eigenvalues can be ordered, for example decreasingly. The map which assigns to the state $[v]$ the collection of the ordered spectra of its one-body reduced density matrices will be denoted by $\Psi: \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{t}_+^*$. This definition can be generalized to an arbitrary compact Lie group K , with \mathfrak{t}_+^* the positive Weyl chamber corresponding to a choice of a Borel subgroup $B \subset G$ of the complexification $G = K^{\mathbb{C}}$.

In the early '80s, the convexity properties of the image of momentum map, $\mu(M)$, were investigated by Atiyah (1982) and Guillemin and Sternberg (1982, 1984) in the case of the abelian group $K = T^n$. They proved that $\mu(M)$ is a convex polytope; it is in fact the convex hull of the image of the set of fixed points. For nonabelian K , this is typically no longer true. However, as it was shown by Guillemin & Sternberg for Kähler manifolds and by Kirwan (1982, 1984) for general symplectic manifolds, the image $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$ is always a convex polytope — the so called *Kirwan polytope*, or *momentum polytope*. Finding an explicit description of this polytope is in general a difficult problem; it is an instance of the quantum marginal problem, or N -representability problem in the case of fermions (Ruskai 1969, Coleman & Yukalov 2000). Following the groundbreaking work of Klyachko (1998) on the famous Horn's problem, which can be formulated as the problem of determining a Kirwan polytope, Berenstein & Sjamaar (2000) found a general solution for the case where M is a coadjoint orbit of a larger group. This was in turn used to solve the one-body quantum marginal problem; namely, sets of inequalities describing the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$ were given (Klyachko 2004, Daftuar & Hayden 2004, Klyachko 2006). Intriguingly, the Kirwan polytope can under certain assumptions be also described in terms of coinvariants or, equivalently, in terms of the representations that occur in the ring of polynomial functions on M (Ness 1984, Brion 1987). In the context of the quantum marginal problem, this has been discovered by Christandl *et al.* (2006, 2007) and Klyachko (2004). The interplay of these two complementary perspectives can also be seen in (Christandl *et al.* 2012), where an algorithm has been given for the more general problem of computing the distribution of eigenvalues of the reduced density matrices of a multipartite pure state in $\mathbb{P}(\mathcal{H})$ drawn at random according to the Haar measure; in particular, this gives an alternative solution to the one-body quantum marginal problem.

The above line of work can also be seen as one of the first successful applications of momentum map geometry to the theory of entanglement (see also Klyachko 2008).

² We assume for simplicity that their Hilbert spaces are of the same dimension.

In 2011, the importance of the momentum map to entanglement was investigated from a different perspective by Sawicki *et al.*. The authors showed that restricting the map Ψ to different local unitary orbits in $\mathbb{P}(\mathcal{H})$ gives rise to a well defined purely geometric measure of entanglement. They also pointed out that the properties of the fibers of Ψ are crucial to the solution of the local unitary equivalence problem and gave an algorithm for checking it for three qubits (Sawicki & Kuś 2011). Geometrically, the problem of local unitary equivalence of two states reduces to determination whether they both belong to the same orbit of the local unitary group $K = SU(\mathcal{H}_1)^{\times N}$. From the physical point of view it corresponds to checking if one of the states can be obtained from the second one by non-dissipative quantum operations restricted to subsystems – an important problem in quantum engineering of states aiming at practical applications (Kraus 2010), (Kraus 2010a).

In the subsequent paper the authors analyzed the geometric structure of the fibers of Ψ for two distinguishable particles, two fermions and two bosons (Huckleberry *et al.* 2012). In all these cases, the K -action is spherical, i.e. the Borel subgroup $B \subset K^{\mathbb{C}}$ has an open orbit in $\mathbb{P}(\mathcal{H})$. By Brion's (1987) theorem, this implies that every fiber of Ψ is contained in a single K -orbit. That is, in these cases every quantum state is up to local unitaries uniquely determined by the spectra of its reduced density matrices. Moreover, each fiber of Ψ has the structure of a symmetric space (Huckleberry *et al.* 2012).

In situations involving larger numbers of particles, e.g., N -qubit systems with $N > 2$, the action of K is not spherical and Brion's theorem cannot be applied. The identification of K -orbits, i.e. classes of states which can be mutually transformed into each other by local unitary transformations, generically requires more information than it is contained in the spectra of the single-particle reduced matrices. The main goal of this paper is to investigate the set of quantum states which are mapped by Ψ to the same point of the Kirwan polytope $\Psi(M)$ (again, these are quantum states whose reduced density matrices have the same spectra) by studying the fibers of Ψ or, equivalently, the symplectic quotients $M_{\alpha} = \Psi^{-1}(\alpha)/K$, in the case where the action of K is no longer spherical. Specifically, we consider the above-mentioned case of N qubits, where the Kirwan polytope is known explicitly (Higuchi *et al.* 2003, Bravyi 2004). A detailed analysis is carried out for three qubits and we make some remarks about the general case. Our main tools are the convexity theorem for projective subvarieties (Ness 1984, Brion 1987), as well as some general properties of orbit spaces which we recall in Section 2. Specifically, we show that M_{α} is generically a two-dimensional stratified symplectic space (Sjamaar & Lerman 1991). For the points in the boundary of the Kirwan polytope the situation is very different. We prove that in this case M_{α} is a single point, i.e. the dimension drops down by two compared with the interior. In particular, these results characterize the K -orbits which are uniquely determined by the spectra of the reduced density matrices. They may be contrasted with the well-known fact that the two-particle reduced density matrices generically suffice to determine a pure-state of three qubits (Linden *et al.* 2002).

The complexified group $G = K^{\mathbb{C}}$ plays its own role in the classification of states of multiparticle quantum systems. Its elements correspond to stochastic local operations with classical communication (SLOCC); see, e.g. Horodecki *et al.* (2009). States can be classified by identifying those which belong to the same SLOCC class, i.e. the same G -orbit. As in the case of locally unitary equivalent states (orbits of K), a detailed analysis of the corresponding Kirwan polytopes is useful in such a classification. In this paper we therefore examine the Kirwan polytopes $\Psi(\overline{G.[v]})$ for the three-qubit SLOCC classes (Dur *et al.* 2000), i.e. for the orbit closures of the complexified group $G = K^{\mathbb{C}}$, and describe their mutual relations. In particular, we find that the map Ψ separates K -orbits when

restricted to the closure of the so-called W -class. That is, states from the W -class are up to local unitaries characterized by the collection of spectra of their one-qubit reduced density matrices.

The paper is organized as follows. In Section 2 we gather useful facts about orbit spaces, especially concerning the principal and maximally-dimensional orbits. In Section 3 we recall the notion of a momentum map and state precisely various versions of the convexity theorems. In Section 4 we discuss in detail the structure of the fibers of Ψ for three qubits. Throughout the paper we prove only new results and otherwise give references to the literature.

2. Qualitative properties of orbit spaces

In this section we gather some useful facts about orbits of group action on a connected manifold M . A good reference is (Bredon 1972).

Let K be a compact connected Lie group acting smoothly on a connected manifold M ($\dim M = n$), i.e. we have a map $\Phi : K \times M \rightarrow M$, $\Phi(g, x) = \Phi_g(x)$, such that

$$\Phi_{gh}(x) = \Phi_g(\Phi_h(x)) \quad \forall x \in M, \quad (2.1)$$

$$\Phi_e(x) = x \quad \forall x \in M.$$

Any orbit $K.x = \Phi(K, x)$ of the K -action is isomorphic to K/K_x , where $K_x \subset K$ is the isotropy subgroup of x , i.e. the closed subgroup consisting of the group elements fixing x . We say that two orbits are of the same type,

$$\text{type}(K/H_1) = \text{type}(K/H_2), \quad (2.2)$$

if and only if H_1 and H_2 are conjugate in K . It is easy to see that (2.2) defines an equivalence relation and hence divides the space of K -orbits into classes. It is also possible to introduce a partial order on this set, and hence on the set of orbit types, by imposing that

$$\text{type}(K/H_1) \geq \text{type}(K/H_2)$$

if and only if H_1 is conjugate to a subgroup of H_2 . Slightly abusing notation, we will write K/H for the corresponding orbit type. The following theorem is an important result for the classification of orbit types (Bredon 1972):

FACT 1 (Principal orbit type). *There exists a single maximal orbit type K/H for the K -action on M , i.e. a subgroup H which is conjugate to subgroups of all other isotropy subgroups. Moreover, the union $M_{(H)}$ of the orbits of type K/H is connected, open and dense in M .*

The subgroup $H \subset K$ described by Theorem 1 is called the principal isotropy subgroup, and the orbits in $M_{(H)}$ are called principal orbits. Other orbits are classified as follows:

- an orbit of type K/H_1 is called singular if and only if $\dim(H_1/H) > 0$.
- an orbit of type K/H_1 is called exceptional if and only if $\dim(H_1/H) = 0$ and H_1/H is a non-trivial finite group.

Let us for the remainder of this article assume that there exists a point $x \in M$ such that the isotropy subgroup K_x is discrete. By Theorem 1, the principal isotropy subgroup is

conjugate to a subgroup of K_x , hence it is also discrete. So if one point $x \in M$ has a discrete isotropy subgroup then there is an open dense connected set of points with a discrete isotropy subgroup. In fact (Montgomery *et al.* 1956):

FACT 2. *The union M_{\max} of the orbits of maximal dimension, i.e. the union of the principal and exceptional orbits, is a connected open dense set whose complement has dimension at most $n - 2$. If in addition M is compact, then there are only finitely many orbit types (i.e. finitely many different isotropy subgroups up to conjugacy).*

The following theorem gives an estimate for the dimension of the sets of singular and exceptional orbits (Montgomery *et al.* 1956):

FACT 3. *Let $r = \min\{\dim K_x : x \in M\}$, i.e. r is the dimension of the principal isotropy subgroup. Let $M_i = \{x \in M : \dim K_x \geq r + i\}$ where $i \geq 1$. Then M_i is a subset of M of codimension at least $i + 1$. If M is compact and $H_1(M, \mathbb{Z}_2) = 0$ then singular and exceptional orbits form a closed set of dimension at most $n - 2$.*

In sections 3 and 4 we will see that the existence of the principal orbit type is crucial to the classification of the fibers of the momentum map.

3. Momentum map

Let K be a compact connected Lie group acting on a symplectic manifold (M, ω) in a symplectic way, i.e. the map Φ_g from (2.1) satisfies the additional condition that $\Phi_g^* \omega = \omega$ for any $g \in K$, where Φ_g^* stands for the pullback of the form by Φ_g . Denote by $\mathcal{F}(M)$ the space of smooth functions on M . The symplectic structure on M induces the structure of a Lie algebra on $\mathcal{F}(M)$, with the Lie bracket given by the Poisson bracket,

$$\{f, g\} = \omega(X_f, X_g),$$

where X_f, X_g are the Hamiltonian vector fields associated to $f, g \in \mathcal{F}(M)$, i.e. $df = \omega(X_f, \cdot)$ and $dg = \omega(X_g, \cdot)$. Hamiltonian actions are defined in such a way that every fundamental vector field arises from a Hamilton function in a consistent way (Guillemin & Sternberg 1990):

DEFINITION 1. *We say that a symplectic action of K on (M, ω) is Hamiltonian if and only if there exists a momentum map $\mu : M \rightarrow \mathfrak{k}^*$, i.e. a map which satisfies the following three conditions:*

1. *For any $\xi \in \mathfrak{k}$, the fundamental vector field $\hat{\xi}(x) = \frac{d}{dt}|_{t=0} \Phi(e^{t\xi}, x)$ is the Hamiltonian vector field for the Hamilton function $\mu_\xi(x) = \langle \mu(x), \xi \rangle$; i.e. $d\mu_\xi = \omega(\hat{\xi}, \cdot)$.*
2. *The induced map $\mathfrak{k} \ni \xi \mapsto \mu_\xi \in \mathcal{F}(M)$ is a homomorphism of Lie algebras, i.e.*

$$\mu_{[\xi_1, \xi_2]}(x) = \{\mu_{\xi_1}, \mu_{\xi_2}\}(x).$$

3. *The map μ is equivariant, i.e. $\mu(\Phi_g(x)) = \text{Ad}_g^* \mu(x)$, where Ad_g^* is the coadjoint action of K on \mathfrak{k}^* defined by $\langle \text{Ad}_g^* \alpha, \xi \rangle = \langle \alpha, \text{Ad}_{g^{-1}} \xi \rangle$ in terms of the adjoint action $\text{Ad}_g \xi = \frac{d}{dt}|_{t=0} g e^{t\xi} g^{-1}$ of K on its Lie algebra \mathfrak{k} .*

For semisimple K , hence in particular for $K = SU(\mathcal{H}_1)^{\times N}$, the momentum map μ is uniquely defined by the above properties (Guillemin & Sternberg 1990).

(a) *Convexity properties of the momentum map*

We will now assume that M is compact and connected. Let us choose a maximal torus $T \subset K$, with Lie algebra \mathfrak{t} , and a positive Weyl chamber $\mathfrak{t}_+^* \subset \mathfrak{t}^*$. Denote by $\Psi : M \rightarrow \mu(M) \cap \mathfrak{t}_+^*$ the map which assigns to $x \in M$ the unique point of intersection $\mu(K.x) \cap \mathfrak{t}_+^*$. Then the following convexity results hold:

1. The image $\Psi(M)$ is a convex polytope, the so-called *Kirwan polytope* (Guillemin & Sternberg (1982, 1984) and Kirwan (1982, 1984)).
2. The fibers of μ (and hence the fibers of Ψ) are connected (Kirwan (1982, 1984)).
3. The map Ψ is an open map onto its image, i.e. for any open subset $U \subset M$ the image $\Psi(U)$ is open in $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$ (Knop 2002).
4. The image $\Psi(M_{\max})$ is convex (Heinzner & Huckleberry 1996).

Denote by $G = K^{\mathbb{C}}$ the complexification of K . This is a complex reductive group; in particular, a choice of positive Weyl chamber \mathfrak{t}_+^* is equivalent to a choice of Borel subgroup $B \subset G$. Let us assume as in (Brion 1987) that $M \subset \mathbb{P}(\mathcal{H})$ is a non-singular G -invariant irreducible subvariety of the complex projective space associated with a rational G -representation \mathcal{H} , and let us choose a K -invariant inner product on \mathcal{H} . Then M is a symplectic manifold when equipped with the restriction of the Fubini–Study form,

$$\omega_{[v]}(\hat{A}_{[v]}, \hat{B}_{[v]}) = 2 \operatorname{Im} \frac{\langle Av | Bv \rangle}{\langle v | v \rangle} = -i \frac{\langle [A, B]v | v \rangle}{\langle v | v \rangle} \quad \forall A, B \in \mathfrak{u}(\mathcal{H}), \quad (3.1)$$

where $[v] \in \mathbb{P}(\mathcal{H})$ is the projection of a vector $v \in \mathcal{H}$, and where \hat{A} is the fundamental vector field generated by the action of $A \in \mathfrak{u}(\mathcal{H})$. More concretely, we can represent the tangent space $T_{[v]}\mathbb{P}(\mathcal{H})$ by $\mathcal{H}/\mathbb{C}v \cong (\mathbb{C}v)^\perp$. In this picture, the tangent vector A_v is given by the orthogonal projection of Av onto $(\mathbb{C}v)^\perp$. M also carries a canonical momentum map $\mu : M \subset \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{t}^*$

$$\langle \mu([v]), A \rangle = -i \frac{\langle v | Av \rangle}{\langle v | v \rangle} \quad \forall A \in \mathfrak{t}, \quad (3.2)$$

In this situation, there are further convexity results for the closures $\overline{G.x}$ and $\overline{B.x}$ of G - and B -orbits, respectively (which in general are singular varieties for which the above results do not apply):

- (v) The image $\Psi(\overline{G.x})$ is a convex polytope (Brion 1987, Ness 1984).
- (vi) The image $\Psi(\overline{B.x})$ is a convex polytope (Guillemin & Sjamaar 2006).
- (vii) There is an open dense set $U \subset \overline{G.x}$ such that $\Psi(\overline{B.y}) = \Psi(\overline{G.x})$ for $y \in U$ (Guillemin & Sjamaar 2006).
- (viii) There is an open dense set $U \subset M$ such that $\Psi(M) = \Psi(\overline{B.x}) = \Psi(\overline{G.x})$ for $x \in U$ (Guillemin & Sjamaar 2006).
- (ix) The collection of different polytopes $\Psi(\overline{B.x})$, where x ranges over M , is finite (Guillemin & Sjamaar 2006).

Since M_{\max} is a connected, open and dense subset of M , (iv) implies the following:

FACT 4. *The set $\mu(M_{\max}) \cap \mathfrak{k}_+^*$ is an open, dense, connected, convex subset of $\Psi(M) = \mu(M) \cap \mathfrak{k}_+^*$. In particular, it contains the (relative) interior of the Kirwan polytope.*

(b) *Fibers of the momentum map*

The following two facts characterizing kernel and image of the differential of the momentum map are well-known consequences of the definition (Guillemin & Sternberg 1982):

FACT 5. *The kernel of $d\mu|_x : T_x M \rightarrow \mathfrak{k}^*$ is equal to the ω -orthogonal complement of $T_x(K.x)$, i.e.*

$$\text{Ker}(d\mu|_x) = \{Y \in T_x M : \omega(\hat{\xi}, Y) = 0 \quad \forall \xi \in \mathfrak{k}\} = (T_x(K.x))^{\perp \omega}.$$

FACT 6. *The image of $d\mu|_x : T_x M \rightarrow \mathfrak{k}^*$ is equal to the annihilator of \mathfrak{k}_x , the Lie algebra of the isotropy subgroup $K_x \subset K$.*

These results have immediate consequences for the characterization of the fibers of μ : Using Fact 6, we observe that $d\mu$ is surjective if and only if the Lie algebra \mathfrak{k}_x is trivial. This in turn happens if and only if the isotropy subgroup K_x is discrete. Therefore, if $d\mu$ is surjective at a single point $x \in M$ (as we have assumed above) then by Fact 2 it is automatically so for all points in $M_{\max} = \{x \in M : \dim K.x \text{ is maximal}\}$. In particular, the implicit function theorem together with Fact 5 implies that for all $x \in M_{\max}$, the tangent space at x of the μ -fiber through x has dimension

$$\dim T_x(\mu^{-1}(\mu(x))) = \dim (T_x(K.x))^{\perp \omega} = \dim M - \dim \mathfrak{k}^* \quad (3.3)$$

and hence:

FACT 7. *For the points $x \in M_{\max}$, i.e. for an open, connected and dense subset of M , the intersection $\mu^{-1}(\mu(x)) \cap M_{\max}$ of the μ -fiber through x with M_{\max} is a $\dim (T_x(K.x))^{\perp \omega}$ -dimensional manifold.*

REMARK 1. *An analogue of (3.3) is true in the case when the principal isotropy subgroup is not discrete, as can be confirmed by using the constant rank theorem instead of implicit function theorem.*

Understanding the fibers $\mu^{-1}(\alpha)$ away from the regular points in M_{\max} is a more delicate problem. Since the fibers of μ are K -invariant, it is convenient to introduce the symplectic quotient $M_\alpha := \mu^{-1}(\alpha)/K_\alpha = \Psi^{-1}(\alpha)/K$, which is in general a stratified symplectic space (Sjamaar & Lerman 1991). It is moreover connected by property (iii) in Subsection a. The following is then immediate:

FACT 8. *The fiber $\mu^{-1}(\alpha)$ intersects a single K -orbit if and only if M_α is a single point or, equivalently, if and only if M_α is zero-dimensional.*

If $\alpha \in \Psi(M_{\max})$ then the maximal-dimensional stratum of M_α is simply $(\mu^{-1}(\alpha) \cap M_{\max})/K_\alpha$ (Meinrenken & Woodward 1999). Since the isotropy subgroup of any point in M_{\max} is discrete,

$$\dim M_\alpha = \dim M - \dim \mathfrak{k}^* - \dim K_\alpha \quad (3.4)$$

In other words, we can by mere dimension counting determine whether a K -orbit in M_{\max} is uniquely determined by its image under the momentum map.

4. Fibers of the momentum map for three qubits

With the momentum map machinery presented in the preceding section we are now well-equipped to analyze when a pure state of three qubits is, up to local unitaries, determined by the spectra of its reduced density matrices.

To this end, let $M = \mathbb{P}(\mathcal{H})$ be the projective space of pure states associated with the three-qubit Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The group of local unitaries $K = SU(2)^{\times 3}$ and its complexification $G = SL(2)^{\times 3}$ act on M by the tensor products. The Lie algebra of K is $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e. triples of traceless antihermitian matrices. As described in the preceding section, M is a symplectic manifold with respect to the Fubini–Study form (3.1) and the K -action is Hamiltonian with canonical momentum map (3.2).

It is easy to see that under the identification of \mathfrak{k}^* with \mathfrak{k} induced by the trace inner product, the image $\mu([v])$ is given by the collection of one-qubit reduced density matrices, namely

$$\mu([v]) = i(\rho_1 - \frac{1}{2}I, \rho_2 - \frac{1}{2}I, \rho_3 - \frac{1}{2}I), \quad (4.1)$$

where I is the 2×2 identity matrix (see, e.g., (Sawicki *et al.* 2011, Christandl *et al.* 2012)).

Let us fix the maximal torus $T \subset K$ to be the set of unitary diagonal matrices with determinant equal to one. Then the Lie algebra \mathfrak{t} is equal to the space of traceless antihermitian diagonal matrices. We choose as the positive Weyl chamber the following set of matrices:

$$\mathfrak{t}_+^* = \{(\text{diag}(-i\lambda_1, i\lambda_1), \text{diag}(-i\lambda_1, i\lambda_1), \text{diag}(-i\lambda_1, i\lambda_1)) : \lambda_i \geq 0\}. \quad (4.2)$$

It follows that, up to some rescaling and shifting, the map Ψ sends a pure state $[v]$ to the (ordered) spectra of its reduced density matrices ρ_1, ρ_2, ρ_3 .

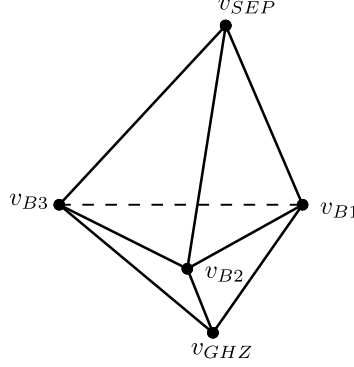
The following theorem describes the Kirwan polytope in terms of inequalities for a general system of N qubits:

FACT 9 ((Higuchi *et al.* 2003)). *For a N -qubit system, the constraints on the one-qubit reduced density matrices ρ_i of a pure state are given by the polygonal inequalities*

$$p_i \leq \sum_{j \neq i} p_j,$$

where $p_i \leq \frac{1}{2}$ denotes the minimal eigenvalue of ρ_i ($i = 1, \dots, N$).

The Kirwan polytope for three qubits is shown in Figure 1.


 Figure 1. The Kirwan polytope $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$ for three qubits.

This polytope has 5 vertices, 9 edges and 6 faces. As a convex set it is of course generated by its vertices, which are

$$\begin{aligned} v_{\text{SEP}} &= \{\text{diag}(0, 1), \text{diag}(0, 1), \text{diag}(0, 1)\}, \\ v_{B1} &= \{\text{diag}(0, 1), \text{diag}(1/2, 1/2), \text{diag}(1/2, 1/2)\}, \\ v_{B2} &= \{\text{diag}(1/2, 1/2), \text{diag}(0, 1), \text{diag}(1/2, 1/2)\}, \\ v_{B3} &= \{\text{diag}(1/2, 1/2), \text{diag}(1/2, 1/2), \text{diag}(0, 1)\}, \\ v_{\text{GHZ}} &= \{\text{diag}(1/2, 1/2), \text{diag}(1/2, 1/2), \text{diag}(1/2, 1/2)\}. \end{aligned} \quad (4.3)$$

Since the polytope is full-dimensional, Sard's theorem (Sternberg 1964) implies the existence of a regular point in M , i.e. a point with discrete isotropy subgroup. Hence:

LEMMA 1. *The set $M_{\max} \subset M$ is connected, open and dense and consists of orbits of dimension $\dim K = 9$.*

We will now analyze points inside the interior of the Kirwan polytope. Notice first that by Fact 4 the preimage of any such point α contains a point $x \in M_{\max}$. Therefore, Fact 7 and (3.3) show that $\mu^{-1}(\alpha) \cap M_{\max}$ is a manifold of dimension

$$\dim(\mu^{-1}(\alpha) \cap M_{\max}) = \dim(T_x K.x)^{\perp \omega} = \dim M - \dim \mathfrak{k}^* = 14 - 9 = 5.$$

Since $K_\alpha = T$ for points in the interior of the positive Weyl chamber, this manifold consists of 3-dimensional T -orbits, and (3.4) implies that

$$\dim M_\alpha = 5 - 3 = 2.$$

If we replace $\mu^{-1}(x)$ by $\Psi^{-1}(x) = K.\mu^{-1}(x)$, we replace each T -orbit by a K -orbit and therefore increase the dimension by

$$\dim \Omega_\alpha = \dim K/K_\alpha = \dim K - \dim T = 6,$$

the dimension of the corresponding coadjoint orbit $\Omega_\alpha = K.\alpha \subset \mathfrak{k}^*$. The general formula for dimension of $\dim \Omega_\alpha$ is

$$\dim \Omega_\alpha = \dim K - \left(\sum_{n=0}^{K_1} m_{1,n}^2 + \sum_{n=0}^{K_2} m_{2,n}^2 + \sum_{n=0}^{K_3} m_{3,n}^2 - 3 \right),$$

where K_i is the number of distinct eigenvalues of ρ_i and $m_{i,n}$ the multiplicity of n -th distinct eigenvalue of ρ_i (Sawicki *et al.* 2011). Summing up, we proved:

THEOREM 1. *For any point α inside the interior of the Kirwan polytope there exists a point $x \in M_{\max}$ such that $\alpha = \mu(x)$. The manifold $\mu^{-1}(\alpha) \cap M_{\max}$ is 5-dimensional, consisting of 3-dimensional T -orbits. Moreover, $\Psi^{-1}(\alpha) \cap M_{\max}$ is an 11-dimensional manifold consisting of 9-dimensional orbits $K.y$ with the property $\mu(K.y) = \Omega_\alpha$, and the symplectic quotient M_α has dimension 2.*

The following is a direct consequence of Theorem 1 (cf. the discussion at the end of Section 3):

COROLLARY 1. *The orbits $K.x$ for which $\mu(K.x) \cap \mathfrak{t}_+^*$ belongs to the interior of Kirwan polytope cannot be separated by the momentum map. In other words, a pure state of three qubits whose spectrum is non-degenerate and satisfies the polygonal inequalities with strict inequality is never determined up to local unitaries by the spectra of its reduced density matrices.*

What is left is to analyze points in the boundary of the Kirwan polytope. We postpone this to the end of the section (see Subsection c).

(a) The SLOCC classes and their Kirwan polytopes

As mentioned in the Introduction, the classification of G -orbits, where $G = K^\mathbb{C}$ gives another view on the entanglement properties of composite system states. In this subsection we explicitly compute the Kirwan polytopes $\Psi(\overline{G.x})$ for all G -orbit closures, and show how they are related to the polytope $\Psi(\mathbb{P}(\mathcal{H}))$. Physically, G -orbits correspond to SLOCC classes (Dur *et al.* 2000), hence our results describe the spectra of the reduced density matrices of the quantum states in each SLOCC entanglement class. The problem of classifying G -orbits in $\mathbb{P}(\mathcal{H})$ is inherently connected to the momentum map geometry as well as to the construction of the so-called Mumford quotient (Kirwan 1984, Ness 1984). We explain this connection in (Sawicki *et al.* 2012, Walter *et al.* 2012).

Classification of G -orbits It has been shown in (Dur *et al.* 2000) that there are six SLOCC entanglement classes, i.e. G -orbits in $\mathbb{P}(\mathcal{H})$. For convenience of the reader we list them below and briefly summarize their basic geometric properties:

1. The G -orbit of the Greenberger–Horne–Zeilinger state, $x_{\text{GHZ}} = [\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)]$ (Greenberger *et al.* 1989): It is the open dense orbit, hence of (real) dimension $\dim \mathbb{P}(\mathcal{H}) = 14$. This also follows from the facts that $\mu(K.x_{\text{GHZ}}) = 0$ and that the orbit $K.x_{\text{GHZ}}$ is Lagrangian, hence of dimension 7, as was shown by Sawicki & Kuś (2011), since for any K -orbit in $\mu^{-1}(0)$ we have the relation $\dim G.x = 2 \cdot \dim K.x$.
2. The G -orbit of the W -state, $x_W = [\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)]$: Here, $\dim G.x_W = 12$, while $\dim K.x_W = 8$. For the proof, it is enough to compute the dimension of the tangent space $T_{x_W}(G.x_W)$, which can be represented as the projection of $\text{Span}_\mathbb{C}\{Ax_W : A \in \mathfrak{g}\}$ onto the orthogonal complement of x_W . The Lie algebra \mathfrak{g} is

equal to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, where

$$\mathfrak{sl}_2(\mathbb{C}) = \text{Span}_{\mathbb{C}} \{E_{12}, E_{21}, E_{11} - E_{22}\},$$

and E_{ij} is the 2×2 matrix with a single non-zero entry equal to one in the i -th row and j -th column. It is easy to see that the following seven vectors

$$\begin{aligned} (E_{12} \otimes I \otimes I)x_W &= (I \otimes E_{12} \otimes I)x_W = (I \otimes I \otimes E_{12})x_W \propto |000\rangle, \\ (E_{21} \otimes I \otimes I)x_W &\propto |110\rangle + |101\rangle, \\ (I \otimes E_{21} \otimes I)x_W &\propto |110\rangle + |011\rangle, \\ (I \otimes I \otimes E_{21})x_W &\propto |101\rangle + |011\rangle, \\ ((E_{11} - E_{22}) \otimes I \otimes I)x_W &\propto -|100\rangle + |010\rangle + |001\rangle, \\ (I \otimes (E_{11} - E_{22}) \otimes I)x_W &\propto +|100\rangle - |010\rangle + |001\rangle, \\ (I \otimes I \otimes (E_{11} - E_{22}))x_W &\propto +|100\rangle + |010\rangle - |001\rangle \end{aligned}$$

span a complex vector space of dimension 6 after projection onto x_W^\perp (the last 3 vectors become linearly dependent). We conclude that $\dim G.x_W = 2 \cdot 6 = 12$. Similarly, one shows that $\dim K.x_W = 8$.

3. The G -orbits through the bi-separable Bell states $x_{B1} = [\frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)] = [|0\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)]$, $x_{B2} = [\frac{1}{\sqrt{2}}(|000\rangle + |101\rangle)]$, and $x_{B3} = [\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)]$: In a similar way as for x_W one can show by an explicit computation that $\dim G.x_{Bk} = 8$ and $\dim K.x_{Bk} = 5$.
4. The G -orbit of separable states, generated by $x_{\text{SEP}} = [|000\rangle]$: By the Kostant–Sternberg theorem, it is the unique symplectic orbit (Sawicki *et al.* 2011). In fact, since $|000\rangle$ is in the K -orbit of the highest weight vector, the Borel–Weil theorem (Sepanski 1995) implies that $G.x_{\text{SEP}} = K.x_{\text{SEP}} \cong K/T \cong G/B$; in particular, the orbit is Kähler and $\dim G.x_{\text{SEP}} = \dim K.x_{\text{SEP}} = 9 - 3 = 6$.

Kirwan polytopes of G -orbit closures Let us write $X_j := \overline{G \cdot x_j}$ for the closures of these G -orbits. The representatives x_j that we have chosen above satisfy the following property:

LEMMA 2. *We have $\Psi(x_j) = \mu(x_j) = v_j$, where v_j is defined in (4.3). Moreover, $\Psi(x_j)$ is the closest point to the origin in the Kirwan polytope $\Psi(X_j)$.*

The first claim follows from a simple computation. The fact that v_j is the closest point to the origin of the Kirwan polytope of the corresponding G -orbit closure follows from the general theory of (Kirwan 1982): the gradient descent with respect to the norm square of the moment map is at any point x implemented by the vector field generated by $i\mu(x) \in i\mathfrak{k} \subseteq \mathfrak{g}$, and one easily checks that $i\widehat{\mu(x_j)}|_{x_j} = 0$. It will be explained in more detail in (Sawicki *et al.* 2012, Walter *et al.* 2012).

We will now describe the Kirwan polytopes of the G -orbit closures: Since $G.x_{\text{GHZ}}$ is dense in $\mathbb{P}(\mathcal{H})$, we immediately obtain that $\Psi(X_{\text{GHZ}})$ is equal to the full Kirwan polytope for three qubits as described by Fact 9. On the other hand, since $G.x_{\text{SEP}} = K.x_{\text{SEP}}$ is a single K -orbit, $\Psi(X_{\text{SEP}})$ is a single point, namely $\Psi(X_{\text{SEP}}) = \{v_{\text{SEP}}\}$. Similarly, one

finds that the Kirwan polytopes for the bi-separable Bell states x_{Bk} are equal to the one-dimensional line segments from v_{Bk} to v_{SEP} . Therefore, the only non-trivial task is the computation of the Kirwan polytope for the W -state:

PROPOSITION 1. *The Kirwan polytope $\Psi(X_W)$ is equal to the convex hull of the points v_{B1} , v_{B2} , v_{B3} and v_{SEP} . In particular, it is of maximal dimension.*

Proof. Clearly, $\Psi(X_W)$ is a subset of the full Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$. On the other hand, Lemma 2 and convexity imply that it is also contained in the half-space through v_W with normal vector v_W . Since the intersection of this half-space with $\Psi(\mathbb{P}(\mathcal{H}))$ is precisely equal to the convex hull of the points v_{B1} , v_{B2} , v_{B3} and v_{SEP} , we only need to show that these points are contained in the Kirwan polytope.

We will in fact show that the corresponding preimages x_j are contained in the orbit closure X_W . For this, we observe that the action of the complexification $T^{\mathbb{C}} \subset G$ of the maximal torus $T \subset K$ applied to x_W gives rise to all states of the form

$$[c_1|100\rangle + c_2|010\rangle + c_3|001\rangle] \quad (c_j \neq 0) \quad (4.4)$$

In particular, $x_j \subseteq X_W = \overline{G \cdot x_W}$ for $j = B1, B2, B3, \text{SEP}$, and the claim follows. \blacksquare

(b) *Sphericity of the W SLOCC class*

In this subsection, we show that $X_W = \overline{G \cdot x_W}$ is a spherical variety. It then follows by Brion's theorem that every quantum state in X_W is, up to local unitaries, characterized by the collection of the spectra of its one-qubit reduced density matrices. In other words, Ψ separates the K -orbits in X_W .

We start by clarifying the geometric structure of X_W . Note that since $G = K^{\mathbb{C}}$ is a complex reductive group, the closures of any orbit closure X_j is a G -invariant irreducible subvarieties of $\mathbb{P}(\mathcal{H})$; however, these varieties will in general be singular. This is in fact already the case for X_W . Indeed, it is known from (Klyachko 2008) that

$$X_W = \mathbb{P}(\mathcal{H}) \setminus (G \cdot x_{\text{GHZ}}) = \{[v] \in \mathbb{P}(\mathcal{H}) : \text{Det}(v) = 0\},$$

where Det is the Cayley hyperdeterminant (the basic invariant for the G -representation \mathcal{H}), and one readily verifies that the tangent space at $x_{\text{SEP}} \in X_W$ has complex dimension $7 > 6 = \dim_{\mathbb{C}} X_W$.

Let us denote by $\mu_W : X_W \rightarrow \mathfrak{k}^*$ the restriction of $\mu : \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{k}^*$ to X_W . Our aim now is to prove that μ_W separates the K -orbits in X_W . To this end we use Brion's theorem (cf. (Huckleberry & Wurzbacher 1990)):

FACT 10. (Brion 1987) *Let $G = K^{\mathbb{C}}$ be a connected complex reductive group, \mathcal{H} a rational G -representation, and X a G -invariant irreducible subvariety of $\mathbb{P}(\mathcal{H})$ (cf. Subsection a). Then the following are equivalent:*

1. X is spherical, i.e. the (every) Borel subgroup B has a Zariski-open orbit in X .
2. For every $x \in X$ the fiber $\mu^{-1}(\mu(x))$ is contained in a single K -orbit, $K \cdot x$.

We will now show that indeed:

PROPOSITION 2. *The G -variety X_W is spherical.*

Proof. Consider the Borel subgroup B , which consists of the lower-triangular matrices in G . Its Lie algebra \mathfrak{b} is equal to $\mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n}$, where

$$\mathfrak{n} = \text{Span}_{\mathbb{C}}\{E_{21} \otimes I \otimes I, I \otimes E_{21} \otimes I, I \otimes I \otimes E_{21}\}.$$

In order to show that X_W is spherical, we have to show that there exists a Zariski-open orbit $B.x$. Since $B.x$ is Zariski-open in its closure, it suffices to show that the closure of $B.x$ is equal to X_W . Now, since the closure of $B.x$ is a closed subvariety, it is either equal to X_W or of lower dimension. Therefore, it suffices to show that the dimension of $B.x$ is equal to the dimension of X_W . Since $B.x$ is a smooth variety, we can compute its dimension by computing the dimension of the tangent spaces at any point. Hence in order to show that X_W is spherical, it suffices to show that $\dim_{\mathbb{C}} T_x(B.x) = \dim_{\mathbb{C}} X_W = 6$ for any single point x in the G -orbit of x_W . We will consider the state

$$x = \left[\frac{1}{\sqrt{4}}(|100\rangle + |010\rangle + |001\rangle + |000\rangle)\right] = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes I \otimes I\right) \cdot x_W.$$

Indeed, one can easily verify that the tangent vectors generated by \mathfrak{b} , i.e. the projection of

$$\begin{aligned} (E_{21} \otimes I \otimes I)x &\propto |110\rangle + |101\rangle + |100\rangle, \\ (I \otimes E_{21} \otimes I)x &\propto |110\rangle + |011\rangle + |010\rangle, \\ (I \otimes I \otimes E_{21})x &\propto |101\rangle + |011\rangle + |001\rangle, \\ ((E_{11} - E_{22}) \otimes I \otimes I)x &\propto -|100\rangle + |010\rangle + |001\rangle + |000\rangle, \\ (I \otimes (E_{11} - E_{22}) \otimes I)x &\propto +|100\rangle - |010\rangle + |001\rangle + |000\rangle, \\ (I \otimes I \otimes (E_{11} - E_{22}))x &\propto +|100\rangle + |010\rangle - |001\rangle + |000\rangle, \end{aligned}$$

onto the orthogonal complement x^{\perp} , span a complex vector space of dimension six. \blacksquare

The following is now a direct consequence of Proposition 2 and Fact 10:

THEOREM 2. *The momentum map μ_W separates all K -orbits inside X_W . That is, every quantum state in the W SLOCC class is (up to local unitaries) uniquely determined by the spectra of its reduced density matrices.*

Points in the interior of $\Psi(X_W)$ are also in the interior of $\Psi(\mathbb{P}(\mathcal{H}))$ and hence for any point α inside the interior $\Psi(X_W) \subset \Psi(\mathbb{P}(\mathcal{H}))$ there is a point $x \in M_{\max}$ such that $\alpha = \mu(x)$. The manifold $\mu^{-1}(\alpha) \cap M_{\max}$ is 5-dimensional. Moreover, $\mu^{-1}(\mu(\Omega_{\alpha})) \cap M_{\max}$ is an 11-dimensional manifold consisting of 9-dimensional orbits $K.y$ with the property $\mu(K.y) = \Omega_{\alpha}$. Every manifold $\mu^{-1}(\mu(\Omega_{\alpha})) \cap M_{\max}$ contains a unique orbit $K.\tilde{x}$ with $\tilde{x} \in G.x_W$. We have illustrated the situation in Figure 2.

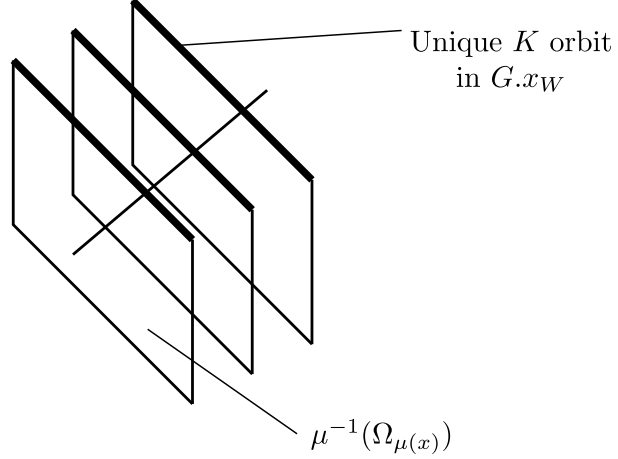


Figure 2. The structure of $\mu^{-1}(\Omega_\alpha)$, where α is in the interior of the polytope $\Psi(X_W)$.

(c) *The boundary of the Kirwan polytope*

In our analysis at the beginning of this section we did only consider quantum states that are mapped into the interior of the Kirwan polytope. We will now prove that any pure quantum state of three qubits that is mapped to the boundary of the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$ is, up to local unitaries, uniquely determined by the spectra of its one-body reduced density matrices.

Let us therefore consider a quantum state $x \in \mathbb{P}(\mathcal{H})$ that is mapped to the boundary of the Kirwan polytope and write

$$\mu(x) = (\text{diag}(-i\lambda_1, i\lambda_1), \text{diag}(-i\lambda_2, i\lambda_2), \text{diag}(-i\lambda_3, i\lambda_3)), \quad (4.5)$$

according to (4.2). Then the minimal eigenvalue p_j of the reduced density matrix ρ_j is given by $p_j = \frac{1}{2} - \lambda_j$. We distinguish two cases:

Non-degenerate case We shall first treat the case where the reduced density matrices of x all have non-degenerate eigenvalue spectrum; that is, $\mu(x)$ is contained in the interior of the positive Weyl chamber. We may assume by symmetry that $\mu(x)$ is contained in the face corresponding to the equation $p_1 = p_2 + p_3$ (cf. Fact 9), i.e.

$$-\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{2}. \quad (4.6)$$

That is, $\mu(x)$ is orthogonal to (annihilated by) the Lie algebra element $i\xi \in \mathfrak{k}$, where

$$\xi = \frac{1}{2} (\text{diag}(1, -1), \text{diag}(1, -1), \text{diag}(1, -1)),$$

and Fact 6 implies that $\mathbb{C}\xi$ is contained in the Lie algebra of the isotropy subgroup of x . It follows that $x = [v]$ for some eigenvector $v \in \mathcal{H}$ of ξ with eigenvalue $1/2$. By diagonalizing the action of ξ on \mathcal{H} we find that

$$x = [c_1|000\rangle + c_2|110\rangle + c_3|101\rangle]$$

for some constants c_j . Any such state is contained in X_W , the closure of the SLOCC class of the W -state, as can be seen by applying $|0\rangle \leftrightarrow |1\rangle$ to the first subsystem and comparing with (4.4). We can therefore use Theorem 2 to conclude that x is determined up to local unitaries by the eigenvalues of its reduced density matrices.

Degenerate case We will now treat the case where at least one of the reduced density matrices is maximally mixed. Without loss of generality, we may assume that $\mu(x)$ is contained in the face of the Kirwan polytope defined by $p_1 = \frac{1}{2}$, i.e. $\lambda_1 = 0$. Note that we *cannot* apply the same reasoning as above, since the faces $\lambda_j = 0$ arise from intersecting $\mu(\mathbb{P}(\mathcal{H}))$ with the positive Weyl chamber and not from the geometry of the momentum map. By the Schmidt decomposition, and up a local unitary on the first subsystem,

$$x = [\frac{1}{\sqrt{2}}(|0\rangle \otimes |\phi\rangle + |1\rangle \otimes |\psi\rangle)]$$

for orthogonal vectors $\langle \phi | \psi \rangle = 0$. The two-qubit reduced density matrix ρ_{23} of any such state is a normalized projector onto the two-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ with basis vectors $|\phi\rangle$ and $|\psi\rangle$; conversely, any other choice of orthonormal basis gives rise to a local unitarily equivalent state.

Let us denote by $\mathbb{G}(2, 4)$ the Grassmannian consisting of two-dimensional subspaces $\mathcal{K} \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$. Similarly to the case of the projective space, we can consider the tensor product action of $G' = SL(2) \times SL(2)$ and of its maximal compact subgroup $K' = SU(2) \times SU(2)$. A momentum map for the K' -action is given by

$$\mu' : \mathbb{G}(2, 4) \ni \mathcal{K} \mapsto i \left(\rho_2 - \frac{1}{2}I, \rho_3 - \frac{1}{2}I \right) \in \mathfrak{k}'^*, \quad \mathfrak{k}' = \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

where ρ_2 and ρ_3 denote the reduced density matrices of $\rho_{23} = \frac{1}{2}P_{\mathcal{K}}$, the normalized projector onto the subspace \mathcal{K} . In view of Fact 10, it suffices to establish sphericity of this Grassmannian with respect to the action of G' :

PROPOSITION 3. *The G' -variety $\mathbb{G}(2, 4)$ is spherical.*

Proof. Let B' denote the Borel subgroup consisting of lower triangular matrices in G' . As in the proof of Proposition 2, we will show that there exists a point $\mathcal{K} \in \mathbb{G}(2, 4)$ at which $\dim_{\mathbb{C}} T_{\mathcal{K}}(B' \cdot \mathcal{K}) = \dim_{\mathbb{C}} \mathbb{G}(2, 4) = 4$ (noting that $\mathbb{G}(2, 4)$ is smooth). It will be convenient to work with coordinates. Let us therefore consider the Plücker embedding

$$\mathbb{G}(2, 4) \rightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^4), \quad \text{Span}_{\mathbb{C}}\{|\phi\rangle, |\psi\rangle\} \mapsto [|\phi\rangle \wedge |\psi\rangle].$$

The image of the subspace $\mathcal{K} = \text{Span}_{\mathbb{C}}\{|++\rangle, |--\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, is

$$x = [|++\rangle \wedge |--\rangle] = [|00\rangle \wedge |01\rangle + |00\rangle \wedge |10\rangle - |01\rangle \wedge |11\rangle - |10\rangle \wedge |11\rangle].$$

It follows that the tangent space $T_{\mathcal{K}}(B' \cdot \mathcal{K})$ is spanned by the vectors

$$\begin{aligned} (E_{21} \otimes I)x &\propto |00\rangle \wedge |11\rangle - |01\rangle \wedge |10\rangle, \\ (I \otimes E_{21})x &\propto |00\rangle \wedge |11\rangle + |01\rangle \wedge |10\rangle, \\ ((E_{11} - E_{22}) \otimes I)x &\propto |00\rangle \wedge |01\rangle + |10\rangle \wedge |11\rangle, \\ (I \otimes (E_{11} - E_{22}))x &\propto |00\rangle \wedge |10\rangle + |01\rangle \wedge |11\rangle, \end{aligned}$$

which are orthogonal to x and linearly independent over \mathbb{C} . We conclude that the tangent space is of complex dimension four. \blacksquare

In summary, we have proved the following result:

THEOREM 3. *Let x be a pure quantum state of three qubits such that $\mu(x)$ is contained in the boundary of the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$ (i.e. its eigenvalues satisfy at least one of the inequalities in Fact 9 with equality). Then x is up to local unitaries uniquely determined by $\mu(x)$, i.e. by the spectra of its one-body reduced density matrices.*

5. Summary

In order to determine which pure states of three qubits are up to local unitaries uniquely determined by the spectra of their reduced density matrices, we have analyzed the change of the structure of the fiber $\Psi^{-1}(\alpha)$ as α varies in the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$. We have shown that $M_\alpha = \Psi^{-1}(\alpha)/K$ is generically a two-dimensional space. For the points in the boundary of the Kirwan polytope, the situation is rather different: We have proved that in this case $\Psi^{-1}(\alpha)/K$ is a single point, i.e. the dimension drops by two compared with the points inside the interior. We have therefore identified all K -orbits that are uniquely determined by the spectra of the reduced density matrices. In addition, we have examined the Kirwan polytopes $\Psi(\overline{G.x_j})$ for all six three-qubit SLOCC classes (i.e. for the closures of the orbits of the complexified group $G = K^{\mathbb{C}}$) and their mutual relation. In particular, we have proved that states from the so-called W SLOCC class are up to local unitaries separated by Ψ , i.e. each K -orbit inside $\overline{G.x_W}$ is characterized by the collection of spectra of the one-qubit reduced density matrices.

Interestingly, the drop of the dimension of M_α on the boundary of the Kirwan polytope has the following counterpart in (Christandl *et al.* 2012): The probability density $f(\alpha)$ of the eigenvalue distribution of the reduced density matrices of a randomly chosen pure state of three qubits vanishes precisely on the boundary of the Kirwan polytope. Since it is well-known that $f(\alpha) = \text{vol } M_\alpha$ for regular points of the momentum map (Duistermaat & Heckman 1982, Meinrenken & Woodward 1999), it is reasonable to wonder whether an analogous statement could hold more generally (the main result of (Meinrenken & Sjamaar 1999) suggests that this might in fact be true).

We also noticed that the polytope $\Psi(X_W)$ associated with the W SLOCC class is of the same dimension as the polytope $\Psi(X_{\text{GHZ}})$ corresponding to the GHZ SLOCC class, whereas for bi-separable and separable states the Kirwan polytope is of strictly lower dimension. On the other hand, it is known that for three qubits only the W and GHZ SLOCC classes represent genuinely entangled states. This intriguing relationship suggests that by looking at the polytopes corresponding to different SLOCC classes one can decide to what extent states from this class are entangled. We believe that this is not a coincidence and conjecture that similar phenomena should be present for N -qubit systems, where $N > 3$.

Acknowledgment

We gratefully acknowledge the support of SFB/TR12 Symmetries and Universality in Mesoscopic Systems program of the Deutsche Forschungsgemeinschaft, ERC Grant QOOLAP, a grant of the Polish

National Science Center under the contract number DEC-2011/01/M/ST2/00379, Polish MNiSW grant no. IP2011048471, the Swiss National Science Foundation (grant PP00P2-128455), the German Science Foundation (grant CH 843/2-1), and the National Center of Competence in Research ‘Quantum Science and Technology’. The authors are especially in debt to Alan Huckleberry for many valuable discussions concerning symplectic and algebraic geometry and for his constant interest in their work. The second author would also like to thank Matthias Christandl, Brent Doran, David Gross, and Eckhard Meinrenken for many fruitful discussions.

References

- Arnold, V. I. 1989 *Mathematical Methods of Classical Mechanics*, Springer-Verlag.
- Atiyah, M. F. 1982 *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14**, 1–15.
- Berenstein, A., Sjamaar, R. 2000 *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. **13**, 433–466.
- Bravyi, S. 2004 *Requirements for compatibility between local and multipartite quantum states*, Quantum Inf. Comput. **4**, 012–026.
- Bredon, G. E. 1972 *Introduction to compact transformation groups*, Pure and Applied Math. 46, Academic Press.
- Brion, M. 1987 *Sur l’image de l’application moment*, Séminaire d’Algèbre Paul Dubreil et Marie-Paule Malliavin, Lecture Notes in Mathematics, vol. 1296, Springer, pp. 177–192.
- Christandl, M., Doran, B., Kousidis, S., Walter, M. 2012 *Eigenvalue Distributions of Reduced Density Matrices*, [arXiv:1204.0741](#).
- Christandl, M., Harrow, A. W., Mitchison, G. 2007 *On Nonzero Kronecker Coefficients and their Consequences for Spectra*, Comm. Math. Phys. **270**, 575–585.
- Christandl, M., Mitchison, G. 2006 *The Spectra of Quantum States and the Kronecker Coefficients of the Symmetric Group*, Comm. Math. Phys. **261**, 789–797.
- Coleman, A. J., Yukalov, V. I. 2000 *Reduced Density Matrices: Coulson’s Challenge*, Lecture Notes in Chemistry, vol. 72, Springer.
- Daftuar, S., Hayden, P. 2004 *Quantum state transformations and the Schubert calculus*, Ann. Phys. **315**, 80–122.
- Duistermaat, J. J., Heckman, G. J. 1982 *On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space*, Invent. Math. **69**, 259–268.
- Dur, W., Vidal, G. & Cirac J. I. 2000 *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A **62**, 062314.
- Greenberger, D. M., Horne, M. A., Zeilinger, A. 1989 *Going Beyond Bell’s Theorem*, in: “Bell’s Theorem, Quantum Theory, and Conceptions of the Universe”, Kafatos M (Ed.), Kluwer, 69–72.
- Guillemin, V., Sjamaar, R. 2006 *Convexity theorems for varieties invariant under a Borel subgroup*, Pure Appl. Math. Q. **2**, 637–653.
- Guillemin, V., Sternberg, S. 1982 *Convexity properties of the moment mapping*, Invent. Math. **67**, 491513.
- Guillemin, V., Sternberg, S. 1984 *Convexity properties of the moment mapping, II*, Invent. Math. **77**, 533546.
- Guillemin, V., Sternberg, S. 1990 *Symplectic Techniques in Physics*, 2nd ed., Cambridge Univ. Press.
- Heinzner, P., Huckleberry, A. 1996 *Kählerian potentials and convexity properties of the moment map*, Invent. Math. **126**, 6584.
- Higuchi, A., Sudbery, A., Szulc, J. 2003 *One-qubit reduced states of a pure many-qubit state: polygon inequalities*, Phys. Rev. Lett. **90**, 107902.
- Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K. 2009 *Quantum entanglement*, Rev. Mod. Phys. **81**, 865–941.
- Huckleberry, A. T., Wurzbacher, T. 1990 *Multiplicity-free complex manifolds*, Math. Ann. **286**, 261–280.
- Huckleberry, A., Kuś, M., Sawicki, A. 2012 *Bipartite entanglement, spherical actions and geometry of local unitary orbits*, [arXiv:1206.4200](#), (submitted).

- Kirwan, F. C. 1982 *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Math. Notes, vol. 31, Princeton Univ. Press.
- Kirwan, F. C. 1984 *Convexity properties of the moment mapping, III*, Invent. Math. **77**, 547552.
- Klyachko, A. 1998 *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4**, 419–445.
- Klyachko, A. 2004 *Quantum marginal problem and representations of the symmetric group*, [arXiv:quant-ph/0409113](#).
- Klyachko, A. 2006 *Quantum marginal problem and N -representability*, J. Phys.: Conf. Series **36**, 72–86.
- Klyachko, A. 2008 *Dynamic Symmetry Approach to Entanglement*, [arXiv:0802.4008](#).
- Knop, F. 2002 *Convexity of Hamiltonian manifolds*, J. Lie Theory **12**, 571–582.
- Kraus, B. 2010 *Local Unitary Equivalence of Multipartite Pure States*, Phys. Rev. Lett. **104**, 020504.
- Kraus, B. 2010a *Local unitary equivalence and entanglement of multipartite pure states*, Phys. Rev. A **82**, 032121.
- Linden, N., Popescu, S., Wootters, W. K. 2002 *Almost Every Pure State of Three Qubits Is Completely Determined by Its Two-Particle Reduced Density Matrices*, Phys. Rev. Lett. **89**, 207901.
- Sjamaar, R., Lerman, E. 1991 *Stratified Symplectic Spaces and Reduction*, Ann. of Math. (2) **134**, 375–422.
- Meinrenken, E., Sjamaar, R. 1999 *Singular reduction and quantization*, Topology **38**, 699–762.
- Meinrenken, E., Woodward, C. 1999 *Moduli spaces of flat connections on 2-manifolds, cobordism, and Witten's volume formulas*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser, pp. 271–295.
- Montgomery, D., Samelson, H., Zippin, L. 1956 *Singular points of a compact transformation group*, Ann. of Math. (2) **63**, 1–9.
- Montgomery, D., Samelson, H., Yang, C. T. 1956 *Exceptional Orbits of Highest Dimension*, Ann. of Math. (2) **64**, 131–141.
- Ness, L. 1984 *A stratification of the null cone via the moment map*, Amer. J. Math. **106**, 12811329 [with an appendix by D. Mumford].
- Ruskai, M. B. 1969 *N -Representability Problem: Conditions on Geminals*, Phys. Rev. **183**, 129–141.
- Sepanski, M. 1995 *Compact Lie groups*, Graduate Texts in Mathematics, vol. 235, Springer.
- Sjamaar, R. 1998 *Convexity properties of the moment mapping re-examined*, Adv. in Math. **138**, 46–91.
- Sawicki, A., Huckleberry, A., Kuś, M. 2011 *Symplectic geometry of entanglement*, Comm. Math. Phys. **305**, 441–468.
- Sawicki, A., Kuś, M. 2011 *Geometry of the local equivalence of states*, J. Phys. A **44**, 495301.
- Sawicki, A., Oszmaniec, M., Kuś, M. 2012 *The convexity property of momentum map, Morse index and Quantum Entanglement* (in preparation).
- Sternberg, S. 1964 *Lectures on differential geometry*, Prentice-Hall.
- Walter, M., Doran, B., Gross, D., Christandl, M. 2012 *in preparation*.